Speed selection mechanism for propagating fronts in reaction-diffusion systems with multiple fields

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We introduce a speed selection mechanism for front propagation in reaction-diffusion systems with multiple fields. This mechanism applies to pulled and pushed fronts alike, and operates by restricting the fields to large *finite* intervals in the comoving frames of reference. The unique velocity for which the center of a monotonic solution for a particular field is insensitive to the location of the ends of the finite interval is the velocity that is physically selected for that field, making thus the solution approximately translation invariant. The fronts for the various fields may propagate at different speeds, all of them being determined though through this mechanism. We present analytic results for the case of piecewise parabolic potentials, and numerical results for other cases.

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I. THE SELECTION MECHANISM

In many systems rendered suddenly unstable, propagating fronts appear. The determination of the speed of a front propagating into an unstable state has attracted a lot of attention, since it cannot be achieved by simply solving an ordinary differential equation in the comoving frame of reference on a one-dimensional infinite domain. Indeed, there are many such solutions on such a domain, even though the propagating front in practice always relaxes to a unique shape and speed. The selection principles that have been formulated to determine the observable front, without having to solve directly the partial differential equation of motion for a range of initial conditions, have involved concepts of linear and nonlinear marginal stability, structural stability, and of causality [1], and all of them try to deal with the puzzle of the reduction of the multiple solutions to the unique observed one. All of these selection principles examine the waves from the viewpoint of the moving front, the corresponding wave equations being reduced then to ordinary differential equations involving the speed v of propagation.

These various approaches can be problematic though in the case of multiple fields, because not all fields need to propagate at the same speed, while the reduction of the set of partial differential equations to a system of ordinary differential equations requires that all fields be functions of the same variable $x-vt$. More recently, a complete analytical understanding of the propagation mechanism and relaxation behavior has emerged for those fronts that are ‘‘pulled along’’ by the spreading of linear perturbations about the unstable state, the so-called ‘‘pulled’’ fronts [2]. This understanding resulted from a detailed study of the relevant partial differential equations and explains fully the behavior of pulled fronts. The speed selection mechanism for those fronts where linear analysis fails, the so-called ‘‘pushed’’ fronts, is still, however, the subject of ongoing research for the case of multiple fields.

The basic problem is that the ordinary differential equations that govern the motion of uniformly translating fronts do not seem to be able to determine the selected velocities for the various fields. Indeed, as we said above, the very existence of different propagation speeds for the various fields makes the examination of the problem from the viewpoint of a particular moving frame of reference seem irrelevant.

Let us take, for example, the equations

$$\frac{\partial \phi_1}{\partial t}=\frac{\partial^2 \phi_1}{\partial x^2}+\phi_1-\phi_1^3,$$

$$\frac{\partial \phi_2}{\partial t}=D\frac{\partial^2 \phi_2}{\partial x^2}+\phi_2-\phi_2^3+K\phi_1, \tag{1}$$

where K is positive. The dynamics of ϕ_1 is always independent of that of ϕ_2, for fields propagating into the unstable state $\phi_1=\phi_2=0$. If $D<1$, both fields move with speed $v=2$. For $D>1$, the ϕ_1 and ϕ_2 fronts propagate with different speeds $v_1=2$ and $v_2=2\sqrt{D}$, respectively [2]. Clearly, the equations indicate that if ϕ_1 is a function of $x-vt$ then ϕ_2 should be too. Both fields should be propagating, therefore, with the same speed, which is the case, however, only for $D<1$. In fact, ϕ_2 always seems to be moving at the maximum available speed.

It would appear thus that the fronts in reaction-diffusion systems with multiple components cannot be properly understood in terms of the properties of the ordinary differential equations that describe uniformly translating solutions. On the other hand, the examination of the full coupled partial differential equations is a rather complicated affair, and there is no universal way for dealing with pushed fronts.

In this paper we present a selection mechanism that applies to fronts invading both unstable and metastable states, whether they be pulled or pushed, and that works even for fields propagating at different speeds. Furthermore, it is easy to apply, since it involves examination of the system from the viewpoint of a single moving frame, resulting thus in coupled ordinary differential equations.

This mechanism is the straightforward generalization of the speed selection principle presented earlier for the case of a single field [3]. It exploits the fact that the observed front of a particular field is translationally invariant in the comoving frame of reference, even on a large *finite* interval, in the sense that its location is effectively independent of the ends of the interval. We shall be solving then the steady state equations of motion on a large finite interval with respect to a reference frame moving at an arbitrary given speed v, subject to the appropriate boundary conditions, obtaining a certain solution for each field. The solution for a particular field, however, will have approximate translational symmetry, thus becoming a physically observable front, only for a certain value of v. It is this value v^* of v that is experimentally observed during the propagation of that field. Thus the selected front is the one that is effectively translationally invariant on a large finite interval, in the comoving frame of reference. Of course, this selected speed v^* will not be appropriate, in general, for the other fields, in the sense that the corresponding solutions for the other fields need not have approximate translational symmetry at that speed.

For values of v different from v^* the midpoint of the field will be either at the left or the right end of the finite interval. It is only at v^* that the midpoint can be anywhere near the center of the interval, becoming in fact indeterminate. Thus a graph of the speed v versus the midpoint of the field will have a *plateau* when v takes the value v^*. The other fields will have such plateaus for other values of v. It is these plateaus, obtained from the graph of v versus the midpoints of the fields, that determine the physically selected speeds.

Let us illustrate our mechanism with an example. We consider the following reaction-diffusion system:

$$\frac{\partial \phi_i}{\partial t}=\frac{\partial^2 \phi_i}{\partial x^2}-\frac{\partial U_i(\phi_i)}{\partial \phi_i}+\sum_{j\neq i} a_{ij}\phi_j, \tag{2}$$

where each U_i is a function of the corresponding ϕ_i only. The fixed points of this system of differential equations provide the appropriate boundary conditions. Let us now assume that all fields have monotonic traveling wave solutions $\phi_i(\xi)$, where $\xi=x-vt$ is the coordinate in a given moving frame of reference, with $v>0$. Clearly, not all of these solutions need to have translational symmetry. The above partial differential equations reduce then to the "steady state" ordinary differential equations

$$\frac{d^2\phi_i}{d\xi^2}+v\frac{d\phi_i}{d\xi}-\frac{\partial U_i(\phi_i)}{\partial \phi_i}+\sum_{j\neq i} a_{ij}\phi_j=0. \tag{3}$$

We solve these equations on a large *finite* interval $[L_1,L_2]$, with $L_1\ll L_2$, subject to the boundary conditions $\phi_j(L_1)=p_j$ and $\phi_j(L_2)=q_j$, say, where j runs over all the fields, and where p_j, q_j, and a_{ij} are constant.

Let us now concentrate on a particular field ϕ_i, and let us find the selected velocity of the corresponding front. Suppose that $\phi_i(\xi)$ is the solution of Eq. (3) subject to the boundary conditions mentioned above. There is only one such solution for a given velocity v. We multiply now Eq. (3) with $d\phi_i/d\xi$ and integrate from L_1 to L_2, obtaining thus

$$v=\frac{U_i(q_i)-U_i(p_i)-\frac{1}{2}w_i^2(L_2)+\frac{1}{2}w_i^2(L_1)-\sum_{j\neq i} a_{ij}\int_{L_1}^{L_2} w_i(\xi)\phi_j(\xi)d\xi}{\int_{L_1}^{L_2} w_i^2(\xi)d\xi}, \tag{4}$$

with $w_i(\xi)=d\phi_i/d\xi$. If $\phi_i(\xi)$ is going to be a physically observable front on this large, but finite, interval, it will have to be essentially translationally invariant. This means that $d\phi_i/d\xi$ will be effectively zero in the regions close to the boundaries, ϕ_i having reached its fixed points there. Consequently $w_i(L_1)$ and $w_i(L_2)$ will tend to zero, while the integrals $\int_{L_1}^{L_2}w_i^2(\xi)d\xi$ and $\int_{L_1}^{L_2}w_i(\xi)\phi_j(\xi)d\xi$ will be finite and practically independent of L_1 and L_2, as $L_1\to-\infty$ and $L_2\to\infty$. Hence the speed v of Eq. (4) becomes independent of the endpoints of the interval, acquiring a unique value v_i^*. In other words, only the front with that particular speed v_i^* can correspond to an essentially translationally invariant field ϕ_i. The other fields will not, in general, have translational invariance at that particular value of v, but that does not affect the above argument. Indeed, these other fields always appear multiplied by the quantity $w_i(\xi)$, which is zero in the regions near the boundaries when v takes the value v_i^* corresponding to a translationally invariant $\phi_i(\xi)$. Thus the integral $\int_{L_1}^{L_2}w_i(\xi)\phi_j(\xi)d\xi$ remains independent of L_1 and L_2, even if these other fields have no translational symmetry.

The requirement that the front ϕ_i be independent of the ends of the finite interval, when $v=v_i^*$, selects therefore the speed

$$v_i^*=\frac{U_i(q_i)-U_i(p_i)-\sum_{j\neq i} a_{ij}\int_{L_1}^{L_2} w_i(\xi)\phi_j(\xi)d\xi}{\int_{L_1}^{L_2} w_i^2(\xi)d\xi}, \tag{5}$$

with $L_1\to-\infty$ and $L_2\to\infty$, as the speed of the physically observed front for the field ϕ_i. Note that no distinction has been made here between metastable and unstable states. Indeed, given *any* particular speed v, we can find a front interpolating between the stable and the unstable or metastable state, provided the solution is found on a finite interval. As the boundaries go to infinity, the value of the speed is restricted to v_i^* and the front becomes the one corresponding to the speed of Eq. (5).

II. ANALYTIC EXAMPLE

We shall demonstrate the proposed selection principle through analytic and numerical work. We shall adopt for our analytic work the following system of dimensionless partial differential equations:

$$\frac{\partial \phi_1}{\partial t} = \frac{\partial^2 \phi_1}{\partial x^2} + f_\mu(\phi_1),$$

$$\frac{\partial \phi_2}{\partial t} = \frac{\partial^2 \phi_2}{\partial x^2} + f_\nu(\phi_2) + g|\phi_1|, \tag{6}$$

where $g>0$, $\mu \geqslant 3$, $\nu \geqslant 3$, and

$$\begin{aligned} f_\mu(u) &= |u| \quad \text{if} \quad |u| \leqslant 1/2, \\ &= \mu(1-|u|) \quad \text{if} \quad |u| \geqslant 1/2. \end{aligned} \tag{7}$$

This piecewise linear choice for the function $f_\mu(u)$ results from a piecewise parabolic potential and will lead to exact analytic solutions. Piecewise linear representations of nonlinearities have often provided an analytically rigorous basis for the study of diffusion systems [4], as well as of nucleation and crystallization problems [5], always on the interval $(-\infty, \infty)$.

We shall be looking for uniformly translating solutions, functions of the variable $\xi = x - vt$, where $v>0$ is an arbitrary given speed. Thus Eqs. (6) become

$$\frac{d^2\phi_1}{d\xi^2} + v\frac{d\phi_1}{d\xi} + f_\mu(\phi_1) = 0, \tag{8}$$

and

$$\frac{d^2\phi_2}{d\xi^2} + v\frac{d\phi_2}{d\xi} + f_\nu(\phi_2) + g|\phi_1| = 0. \tag{9}$$

The mirror symmetries of Eqs. (8) and (9) allow us to work with positive fields only. So we shall assume, without loss of generality, that $\phi_1 \geqslant 0$, $\phi_2 \geqslant 0$. When the fields are constant in space and time, they are at their fixed points. These fixed points (ϕ_1, ϕ_2) are the points (0,0) (unstable fixed point), (0,1) (saddle point), and $(1, 1+g/\nu)$ (stable fixed point). We shall be interested in fronts invading the unstable state $\phi_1 = 0$, $\phi_2 = 0$, so we need to solve Eqs. (8) and (9) on the *finite* interval $[L_1, L_2]$, subject to the boundary conditions $\phi_1(L_1) = 1$, $\phi_2(L_1) = 1 + g/\nu$, $\phi_1(L_2) = 0$, $\phi_2(L_2) = 0$, where $L_1 \ll 0 \ll L_2$. These boundary conditions ensure that the system makes a phase transition from the unstable state to the stable state. Furthermore, we shall define the midpoints ξ_1 and ξ_2 of the fields ϕ_1 and ϕ_2 through the relations $\phi_1(\xi_1) = \frac{1}{2}$ and $\phi_2(\xi_2) = \frac{1}{2}$, respectively, noting that the fields and their slopes have to be continuous at these points.

The dynamics of ϕ_1 is decoupled from the dynamics of ϕ_2, consequently we can easily find the corresponding solution. There are five boundary conditions that must be satisfied, namely two at the edges, the continuity of ϕ_1 and of $d\phi_1/d\xi$ at ξ_1, and the definition of ξ_1. On the other hand, the solution of Eq. (8) for the field ϕ_1 will involve five unknown constants for any given value of v, namely ξ_1 and two constants for each linear piece of $f_\mu(\phi_1)$. We expect, therefore, a unique solution $\phi_1(\xi)$ for each value of v.

Indeed, the exact solution of Eq. (8) for the field ϕ_1 is

$$\begin{aligned} \phi_1(\xi) &= 1 - \frac{e^{m_1(\xi - L_1)} - e^{m_2(\xi - L_1)}}{2e^{m_1(\xi_1 - L_1)} - 2e^{m_2(\xi_1 - L_1)}} \quad \text{if} \quad L_1 \leqslant \xi \leqslant \xi_1 \\ &= \frac{e^{k_1(\xi - L_2)} - e^{k_2(\xi - L_2)}}{2e^{k_1(\xi_1 - L_2)} - 2e^{k_2(\xi_1 - L_2)}} \quad \text{if} \quad \xi_1 \leqslant \xi \leqslant L_2, \end{aligned} \tag{10}$$

where

$$k_1 = \tfrac{1}{2}(-v + \sqrt{v^2 - 4}), \tag{11}$$

$$k_2 = \tfrac{1}{2}(-v - \sqrt{v^2 - 4}), \tag{12}$$

$$m_1 = \tfrac{1}{2}(-v + \sqrt{v^2 + 4\mu}), \tag{13}$$

$$m_2 = \tfrac{1}{2}(-v - \sqrt{v^2 + 4\mu}), \tag{14}$$

and ξ_1 satisfies

$$\frac{k_1 e^{k_1(\xi_1 - L_2)} - k_2 e^{k_2(\xi_1 - L_2)}}{e^{k_1(\xi_1 - L_2)} - e^{k_2(\xi_1 - L_2)}} = -\frac{m_1 e^{m_1(\xi_1 - L_1)} - m_2 e^{m_2(\xi_1 - L_1)}}{e^{m_1(\xi_1 - L_1)} - e^{m_2(\xi_1 - L_1)}}. \tag{15}$$

The solution of Eq. (15) gives ξ_1 as a function of the speed v. We note that $m_1 > 0 > m_2$ and $k_2 < k_1 < 0$. For a given value of v, Eqs. (10)–(15) determine fully the field $\phi_1(\xi)$.

As shown in our earlier work [3], the graph of v versus ξ_1 has a plateau at $v = v_{c1}$, where $v_{c1} = (\mu+1)/\sqrt{2\mu - 2}$. Indeed, if we require $L_1 \ll \xi_1 \ll L_2$, then Eq. (15) reduces to $k_2 + m_1 = 0$. This equation has a real solution, $v = v_{c1}$, provided $\mu \geqslant 3$. In other words, for that particular value of v the midpoint ξ_1 can be anywhere in the interval and cannot be determined, rendering thus the front effectively translationally invariant on the finite domain $[L_1, L_2]$. The value v_{c1} is therefore the selected speed of ϕ_1 if $\mu \geqslant 3$ (pushed case). If

$v > v_{c1}$, then ξ_1 is close to L_1, while for $v < v_{c1}$ we find that ξ_1 lies close to L_2. The existence of the plateau at v_{c1} shows thus that the only value of v for which the field ϕ_1 is approximately translationally invariant is v_{c1}.

Let us now turn our attention to the field ϕ_2. The relevant equation must be solved on the three segments determined by L_1, L_2, ξ_1, and ξ_2. There are seven boundary conditions: two at the edges, four from the continuity of ϕ_2 and $d\phi_2/d\xi$ at the points ξ_1 and ξ_2, and the definition of ξ_2. There are also seven unknown quantities for a given value of v, two on each of the three segments, and one on ξ_2. Note that Eq. (15) determines the other midpoint. We expect therefore a unique solution $\phi_2(\xi)$ for each given value of v.

Clearly, if g were zero, then the speed of propagation for ϕ_2 would be $v_{c2} = (\nu+1)/\sqrt{2\nu-2}$, and the two fields would be completely decoupled. We need to examine what happens for $g \neq 0$. We can obtain some qualitative results by looking at Eqs. (8) and (9). We distinguish two cases.

A. Case $v_{c2} > v_{c1}$

(i) We examine the case $v > v_{c2} > v_{c1}$ first. Since $v > v_{c1}$, we have $\xi_1 \approx L_1$, as discussed in our earlier work [3]. Therefore, the field ϕ_1 falls very rapidly from 1 to 0, and it remains equal to 0 on almost all of the interval $[L_1, L_2]$. Since ϕ_1 is approximately 0 almost everywhere, Eqs. (8) and (9) decouple. Thus ϕ_2 behaves as if g were equal to 0. Since $v > v_{c2}$, this implies that $\xi_2 \approx L_1$. Hence ϕ_2 is also 0 practically everywhere. Both fields are essentially on the fixed point (0,0).

(ii) We examine the case $v_{c2} > v > v_{c1}$ next. Again $\xi_1 \approx L_1$, since $v > v_{c1}$, hence ϕ_1 is essentially 0 on almost all of the interval. The fields decouple once more, but now $\xi_2 \approx L_2$, since $v < v_{c2}$. Hence, the field ϕ_2 is nonzero up to the point $\xi = L_2$. The only fixed point that is available for the two fields is then the point (0,1). Thus ϕ_1 starts out at L_1 having the value 1 and very rapidly drops down to 0. The field ϕ_2, on the other hand, starts out having the value $1 + g/\nu$ at L_1, drops down to the value 1 almost immediately, and then it stays there till it reaches the other edge, where it drops down to 0. Thus the midpoint of ϕ_2 shifts abruptly from the left edge to the right edge the very moment we pass from case (i) to case (ii), i.e., at $v = v_{c2}$, because $\xi_2 \approx L_1$ for $v > v_{c2}$, but $\xi_2 \approx L_2$ for $v < v_{c2}$.

(iii) We examine finally the case $v_{c2} > v_{c1} > v$. Since $v_{c1} > v$, we have $\xi_1 \approx L_2$. Therefore ϕ_1 remains on the value 1 on almost all of the interval, dropping down to 0 only very close to the right edge. The only available fixed point for the two fields is then the point $(1,1+g/\nu)$. Therefore, the field ϕ_2 remains stuck at the value $1 + g/\nu$ almost everywhere, dropping to 0 only very close to the right edge, whereby $\xi_2 \approx L_2$. We note that the abrupt shift of the midpoint of ϕ_1 occurs at $v = v_{c1}$.

These arguments indicate then that the field ϕ_1 acquires approximate translational invariance when $v = v_{c1}$, as expected, since its dynamics is decoupled from the dynamics of ϕ_2. At that speed we have a plateau of v versus ξ_1. On the other hand, the plateau of v versus ξ_2 occurs at $v = v_{c2}$. Therefore the front of ϕ_2 propagates with the speed v_{c2}. Indeed then, our finite interval mechanism gives the selected velocities for both fields, in spite of their being different. We also note that in all likelihood $\xi_1 < \xi_2$, since in the region $v_{c1} < v < v_{c2}$ we found that $\xi_1 \approx L_1$ and $\xi_2 \approx L_2$.

B. Case $v_{c2} < v_{c1}$

(i) We examine the case $v > v_{c1} > v_{c2}$ first. Since $v > v_{c1}$, we have $\xi_1 \approx L_1$, as discussed in our earlier work [3]. Therefore the field ϕ_1 falls very rapidly from 1 to 0, and it remains equal to 0 on almost all of the interval $[L_1, \quad L_2]$. Since ϕ_1 is approximately 0 almost everywhere, Eqs. (8) and (9) decouple. Thus ϕ_2 behaves as if g were equal to 0. Since $v > v_{c2}$, this implies that $\xi_2 \approx L_1$. Hence ϕ_2 is also 0 practically everywhere. Both fields are essentially on the fixed point (0,0).

(ii) We examine the case $v_{c1} > v > v_{c2}$ next. Now $\xi_1 \approx L_2$, since $v < v_{c1}$, hence ϕ_1 is stuck on the value 1 on almost all of the interval. The only fixed point that is available for the two fields then is the point $(1,1+g/\nu)$. That means that ϕ_2 must be stuck at the value $1 + g/\nu$ on almost all of the interval, dropping down to 0 only close to the right edge. Hence $\xi_2 \approx L_2$. Thus both the midpoints of ϕ_2 and ϕ_1 shift suddenly from the left edge to the right edge the very moment we pass from case (i) to case (ii), i.e., at $v = v_{c1}$.

(iii) We examine finally the case $v_{c1} > v_{c2} > v$. Since $v_{c1} > v$, we have $\xi_1 \approx L_2$. Therefore ϕ_1 remains on the value 1 on almost all of the interval, dropping down to 0 only very close to the right edge. The only available fixed point for the two fields is once more the point $(1,1+g/\nu)$. Therefore, the field ϕ_2 remains stuck at the value $1 + g/\nu$ almost everywhere, dropping to 0 only very close to the right edge, whereby $\xi_2 \approx L_2$.

These arguments indicate then that the field ϕ_1 acquires approximate translational invariance when $v = v_{c1}$, since its dynamics is decoupled from the dynamics of ϕ_2. At that speed we have a plateau of v versus ξ_1. On the other hand, the plateau of v versus ξ_2 occurs also at $v = v_{c1}$. Therefore the front of ϕ_2 propagates with the speed v_{c1}. In this case both fields propagate at the same speed.

We note that the field ϕ_2 always propagates at the maximum possible speed, i.e., v_{c2} in case A and v_{c1} in case B, just as the fields of Eqs. (1). Our finite interval mechanism is able to handle both cases though. We also note that the speeds of propagation are independent of the coupling constant g, irrespective of how large or small it is.

Let us now verify this behavior by solving analytically Eq. (9) to find ϕ_2, given the solution of Eq. (10) for the field ϕ_1.

We shall assume that $\xi_1 \leqslant \xi_2$, for the sake of definiteness. This situation is appropriate for the case $v_{c2} > v_{c1}$, according to the arguments presented above. If contradictions arise due to this assumption, it will be easy enough to repeat the work with the contrary assumption. In fact, it turns out that the relation $\xi_1 \leqslant \xi_2$ holds even if $v_{c2} < v_{c1}$, in the examples we shall present.

Let us examine the region $\xi_2 \leqslant \xi \leqslant L_2$ first. The boundary conditions $\phi_2(\xi_2) = \frac{1}{2}$ and $\phi_2(L_2) = 0$ determine the two constants that will appear in the solution of the ordinary differential equation on this interval. Thus the full solution for ϕ_2 on the interval $[\xi_2,L_2]$ turns out to be

$$\phi_2(\xi)=(\xi-L_2)\left(-\frac{e^{k_1(\xi-L_2)}}{2k_1+v}+\frac{e^{k_2(\xi-L_2)}}{2k_2+v}\right)z_1 + \gamma(e^{k_1(\xi-L_2)}-e^{k_2(\xi-L_2)}), \quad (16)$$

where

$$z_1=g/(2e^{k_1(\xi_1-L_2)}-2e^{k_2(\xi_1-L_2)}), \quad (17)$$

and

$$\gamma=\frac{\frac{1}{2}-(\xi_2-L_2)[-e^{k_1(\xi_2-L_2)}/(2k_1+v)+e^{k_2(\xi_2-L_2)}/(2k_2+v)]z_1}{e^{k_1(\xi_2-L_2)}-e^{k_2(\xi_2-L_2)}}. \quad (18)$$

Equation (16) now yields the quantity $\phi_2'(\xi_2)$,

$$\phi_2'(\xi_2)=\left(-\frac{e^{k_1(\xi_2-L_2)}}{2k_1+v}+\frac{e^{k_2(\xi_2-L_2)}}{2k_2+v}\right)z_1 + \frac{2z_1(\xi_2-L_2)}{e^{-k_2(\xi_2-L_2)}-e^{-k_1(\xi_2-L_2)}} + \frac{k_1e^{k_1(\xi_2-L_2)}-k_2e^{k_2(\xi_2-L_2)}}{2e^{k_1(\xi_2-L_2)}-2e^{k_2(\xi_2-L_2)}}. \quad (19)$$

We can now use the known values of $\phi_2(\xi_2)$ and $\phi_2'(\xi_2)$ as boundary conditions in order to solve Eq. (9) on the interval $[\xi_1,\xi_2]$. We find

$$\phi_2(\xi)=1+\frac{z_1(e^{k_1(\xi-L_2)}-e^{k_2(\xi-L_2)})}{\nu+1}+Ae^{n_1(\xi-L_2)} + Be^{n_2(\xi-L_2)}, \quad (20)$$

where

$$n_1=\tfrac{1}{2}(-v+\sqrt{v^2+4\nu}), \quad (21)$$

$$n_2=\tfrac{1}{2}(-v-\sqrt{v^2+4\nu}), \quad (22)$$

with $n_1>0$ and $n_2<0$,

$$\Omega_1=-\frac{1}{2}-\frac{z_1}{\nu+1}(e^{k_1(\xi_2-L_2)}-e^{k_2(\xi_2-L_2)}), \quad (23)$$

$$\Omega_2=\phi_2'(\xi_2)-\frac{z_1}{\nu+1}(k_1e^{k_1(\xi_2-L_2)}-k_2e^{k_2(\xi_2-L_2)}), \quad (24)$$

$$A=\exp[-n_1(\xi_2-L_2)]\frac{\Omega_2-n_2\Omega_1}{n_1-n_2}, \quad (25)$$

$$B=\exp[-n_2(\xi_2-L_2)]\frac{\Omega_2-n_1\Omega_1}{n_2-n_1}. \quad (26)$$

We can now use Eq. (20) to find $\phi_2(\xi_1)$ and $\phi_2'(\xi_1)$. We get

$$\phi_2(\xi_1)=1+\frac{g}{2+2\nu}+Ae^{n_1(\xi_1-L_2)}+Be^{n_2(\xi_1-L_2)} \quad (27)$$

and

$$\phi_2'(\xi_1)=(k_1e^{k_1(\xi_1-L_2)}-k_2e^{k_2(\xi_1-L_2)})\frac{z_1}{\nu+1}+n_1Ae^{n_1(\xi_1-L_2)} + n_2Be^{n_2(\xi_1-L_2)}. \quad (28)$$

Finally, we can use these values of ϕ_2 and ϕ_2' at ξ_1 in order to find the solution to ϕ_2 in the interval $[L_1,\xi_1]$. Imposing then the boundary condition $\phi_2(L_1)=1+g/\nu$ on this solution leads to the final relation

$$\begin{aligned}\Omega_2-n_1\Omega_1 &= \left(\frac{g}{2\mu-2\nu}+\frac{g}{2+2\nu}\right)\frac{k_1e^{k_1(\xi_1-L_2)}-k_2e^{k_2(\xi_1-L_2)}}{e^{k_1(\xi_1-L_2)}-e^{k_2(\xi_1-L_2)}}\\ &\quad\times(\exp[-(n_1-n_2)(\xi_1-L_1)]-1)e^{n_2(\xi_2-\xi_1)}\\ &\quad+(\Omega_2-n_2\Omega_1)\exp[(n_1-n_2)(-\xi_2+L_1)]\\ &\quad+\left(\frac{g}{\nu}+\frac{g}{2\mu-2\nu}-\frac{g}{2+2\nu}\right)(n_2\exp[-(n_1-n_2)\\ &\quad\times(\xi_1-L_1)]-n_1)e^{n_2(\xi_2-\xi_1)}. \end{aligned} \quad (29)$$

We can use this equation to find the plateau of ϕ_2.

Indeed, let us take the case $v_{c2}>v_{c1}$ first. If $v>v_{c1}$, then we must have $\xi_1\approx L_1$. In that case we can show that any ξ_2 that is far from L_1 and L_2 will satisfy Eq. (29), provided $k_2+n_1=0$, a relation equivalent to the requirement that v be equal to v_{c2}. Thus ϕ_2 has a plateau above v_{c1}, at the speed $v=v_{c2}$. Note also that the midpoint ξ_2 of ϕ_2 is already at L_2 when the midpoint of ϕ_1 shifts to L_2.

On the other hand, if we look at the case $v_{c2}<v_{c1}$, then we see that below v_{c1} we must have $\xi_1\approx L_2$. But since all the above analytic equations have been derived under the assumption $\xi_1\leqslant\xi_2$, we conclude that ξ_2 must be close to L_2 as well. Thus, when the midpoint ξ_1 of ϕ_1 shifts abruptly to the right edge, it forces the midpoint of the other field to go there as well, provided the analytic equations have solutions con-

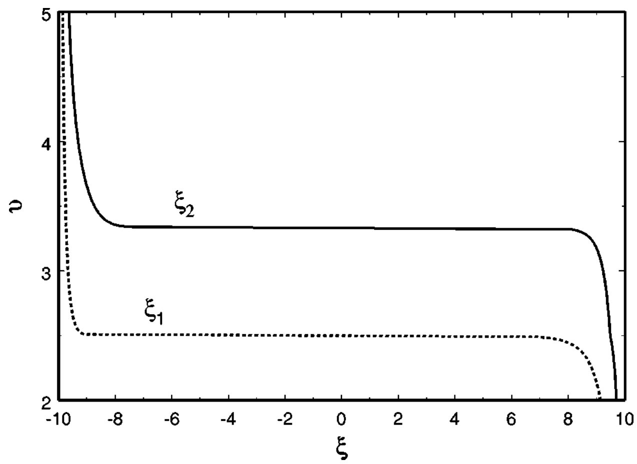

FIG. 1. The speed v as a function of the midpoints of the fronts invading the unstable state, for the system of Eqs. (8) and (9), with $L_1=-10$, $L_2=10$, $g=9.5$, $\mu=9$, and $\nu=19$. The plateau is at $v=2.5$ for ϕ_1, and at $v=\frac{10}{3}$ for ϕ_2. All quantities are dimensionless.

sistent with the assumption $\xi_1<\xi_2$. The corresponding behavior is illustrated by the examples of Figs. 1 and 2, verifying thus the qualitative conclusions drawn earlier. In particular, these figures confirm the assumption $\xi_1<\xi_2$.

We see then that, if $v_{c2}>v_{c1}$, the field ϕ_2 has a plateau at the highest speed v_{c2}. If, on the other hand, $v_{c2}<v_{c1}$, then ϕ_1 pulls ϕ_2 and forces it to propagate at the higher speed v_{c1}. This behavior is seen even when the coupling constant g takes very small values, and matches the behavior of the fields that obey Eq. (1).

III. NUMERICAL EXAMPLES

We can demonstrate our selection mechanism numerically as well. Let us examine the following system:

$$\frac{\partial \phi_1}{\partial t}=\frac{\partial^2 \phi_1}{\partial x^2}+h_{b_1}(\phi_1),$$

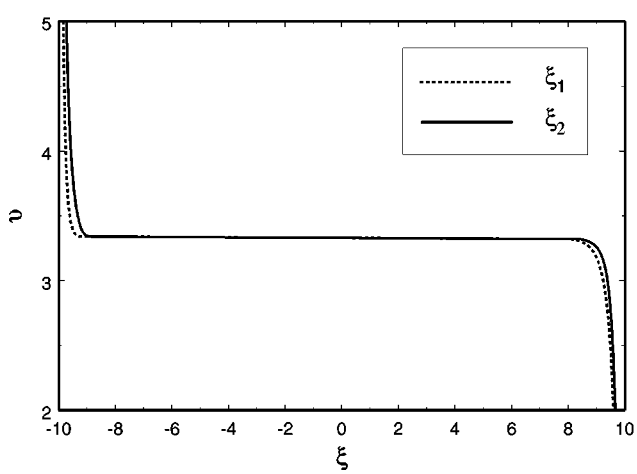

FIG. 2. The speed v as a function of the midpoints of the fronts invading the unstable state, for the system of Eqs. (8) and (9), with $L_1=-10$, $L_2=10$, $g=4.5$, $\mu=19$, and $\nu=9$. The plateau is at $v=\frac{10}{3}$ for both ϕ_1 and ϕ_2. All quantities are dimensionless.

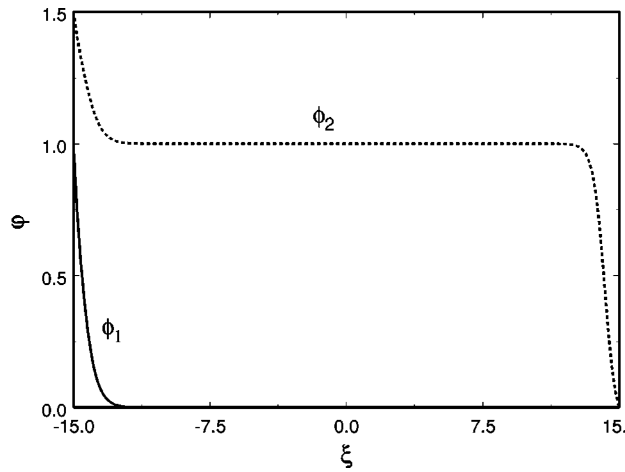

FIG. 3. The solutions of Eqs. (30) that interpolate between the fixed points (0,0) and (1,1.5), for $L_1=-15$, $L_2=15$, $v=2.7$, $g=21$, $b_1=\frac{1}{8}$, and $b_2=\frac{1}{18}$. These solutions lie on the saddle point (0,1) on most of the interval.

$$\frac{\partial \phi_2}{\partial t}=\frac{\partial^2 \phi_2}{\partial x^2}+h_{b_2}(\phi_2)+g\phi_1, \tag{30}$$

where

$$h_b(u)=\frac{1}{b}u(1-u)(b+u). \tag{31}$$

It was this particular choice of $h(u)$ that was used for the case of a single field when the concepts of linear and nonlinear marginal stability were first proposed [6]. That study found that for $0<b<\frac{1}{2}$ the selected speed of the single field for the front invading the unstable state is $(2b+1)/\sqrt{2b}$.

We shall consider values of b less than $\frac{1}{2}$ (pushed case). Thus, if the coupling constant g were 0, then the two fields ϕ_1 and ϕ_2 of Eq. (30) would propagate separately, with speeds $v_{c1}=(2b_1+1)/\sqrt{2b_1}$ and $v_{c2}=(2b_2+1)/\sqrt{2b_2}$, respectively.

We have solved Eqs. (30) numerically on a finite ξ domain for the $h(u)$ of Eq. (31), assuming that both fields are functions of the variable $\xi=x-vt$, with $g=21$, $b_1=\frac{1}{8}$, and $b_2=\frac{1}{18}$, subject to the boundary conditions $\phi_1(L_1)=1$, $\phi_1(L_2)=0$, $\phi_2(L_2)=0$, and $\phi_2(L_1)=1.5$. These values correspond to the stable and unstable fixed points (1,1.5) and (0,0) for the fields ϕ_1 and ϕ_2, the points in other words where Eqs. (30) acquire uniform solutions. The solutions that interpolate between these two fixed points are shown in Fig. 3. We can see that the two fields are at the saddle point $\phi_1=0$, $\phi_2=1$, on almost all of the interval. This feature reminds us of the dual fronts, where the decomposition from the unstable to the stable state proceeds via an intermediate saddle point, with the two fields propagating at different speeds [7].

Figure 4 shows the locations ξ_1 and ξ_2 where the fields ϕ_1 and ϕ_2 attain the value $\frac{1}{2}$, at a given arbitrary speed v, when $g=21$, $b_1=\frac{1}{8}$, and $b_2=\frac{1}{18}$. We can deduce from this figure that the field ϕ_1 propagates with speed $v_{c1}=2.5$,

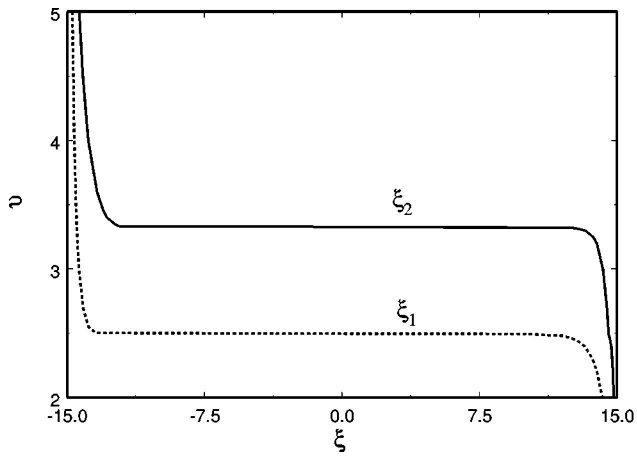

FIG. 4. The speed v as a function of the midpoints of the fronts invading the unstable state, for the system of Eqs. (30), with $L_1 = -15$, $L_2=15$, $g=21$, $b_1=\frac{1}{8}$, and $b_2=\frac{1}{18}$. The plateau is at $v=2.5$ for ϕ_1 and at $v=\frac{10}{3}$ for ϕ_2. All quantities are dimensionless.

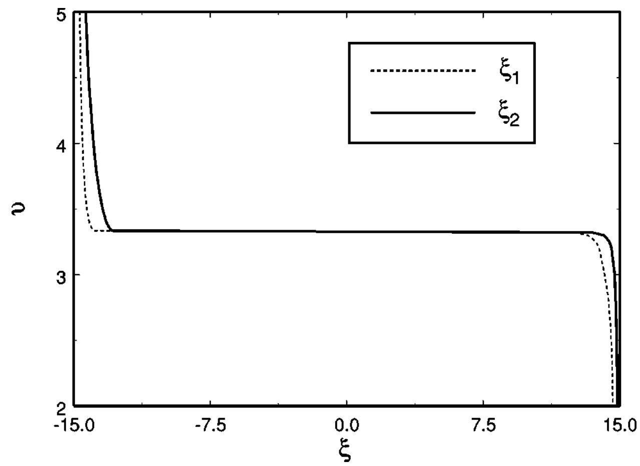

FIG. 5. The speed v as a function of the midpoints of the fronts invading the unstable state, for the system of Eqs. (30), with $L_1 = -15$, $L_2=15$, $g=21$, $b_1=\frac{1}{18}$, and $b_2=\frac{1}{8}$. The plateau is at $v=\frac{10}{3}$ for both ϕ_1 and ϕ_2. All quantities are dimensionless.

while the field ϕ_2 propagates with speed $v_{c2}=\frac{10}{3}$. Indeed, we see that at these velocities there are plateaus of v versus the location where each field acquires the value $\frac{1}{2}$. In other words, the location of the midpoint of the front for the field ϕ_1 or ϕ_2 is indeterminate at the corresponding velocity v_{c1} or v_{c2}, rendering the solution essentially translation invariant there. We note that the two fronts propagate at different speeds, and that $\xi_1 < \xi_2$.

We have also solved Eqs. (30) for the case $g=21$, $b_1=\frac{1}{18}$, and $b_2=\frac{1}{8}$. The unstable fixed point is still the point $\phi_1=\phi_2=0$, but the stable fixed point is the point $\phi_1=1$, $\phi_2=1.7768$, since these values satisfy Eqs. (30). Hence the solutions, we seek, have to interpolate between these two fixed points, subject to the boundary conditions $\phi_1(L_1)=1$, $\phi_1(L_2)=0$, $\phi_2(L_2)=0$, and $\phi_2(L_1)=1.7768$. Figure 5 shows the midpoints ξ_1 and ξ_2 of the two fields at an arbitrary speed v. We observe once more that there are the usual plateaus, ξ_1 being again less than ξ_2. However, both plateaus occur at the speed $v=v_{c1}=\frac{10}{3}$. The field ϕ_2 is pulled by ϕ_1 and is forced to propagate at v_{c1}, rather than at its own lower speed $v_{c2}=2.5$. The existence of the common plateau indicates once again that at that particular speed v_{c1} the locations of the midpoints of the fields become indeterminate, making thus the two fields effectively translation invariant.

IV. CONCLUDING REMARKS

We see then that requiring the solution of a field to have approximate translational invariance on a finite interval in the comoving frame of reference results in the selection of a speed for the front. We can adopt then a selection principle that reads ‘‘the selected front is the one that is approximately translationally invariant on a large finite interval, with respect to the comoving frame of reference.’’ This principle is very easy to implement, especially numerically. Indeed, it suffices to solve the moving frame equation on a large finite interval, for an arbitrary propagation speed. For large speeds we expect the midpoint of the front to be close to the left boundary. As the speed is lowered, the midpoint suddenly moves to the right boundary. The speed v^* at which this sudden move occurs is the speed selected by the physically observed front.

Of course, the selection of a certain plateau for a particular field need not include the selection of such a plateau for other fields as well. In fact, the advantage of our mechanism is that the graph of the speed of propagation versus the midpoint of each field allows us to find the plateaus for all the fields, and hence the corresponding physically selected velocities. The approximate translational invariance that each field acquires as its plateau is reached need not involve the other fields as well.

Thus our mechanism can find the selected velocities for all the fields present in a system of coupled partial differential equations, by simply solving a much simpler system of coupled ordinary differential equations, having assumed that all the fields are traveling at the same arbitrary speed v. The value of v for which the midpoint of a particular field becomes indeterminate is the physically selected velocity for that field.

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